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Two loop Kähler potential in β-deformed $\mathcal{N} = 4$ SYM theory

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Abstract: In $\mathcal{N} = 2$ superconformal field theories the Kähler potential is known to be tree level exact. The β-deformation of $\mathcal{N} = 4$ SU(N) SYM reduces the amount of supersymmetry to $\mathcal{N} = 1$, allowing for non-trivial, superconformal loop corrections to the Kähler potential. We analyse the two-loop corrections on the Coulomb branch for a complex deformation. For an arbitrary chiral field in the Cartan subalgebra we reduce the problem of computing the two-loop Kähler potential to that of diagonalising the mass matrix, we then present the result in a manifestly superconformal form. The mass matrix diagonalisation is performed for the case of the chiral background that induces the breaking pattern $\mathrm{SU}(N) \to \mathrm{SU}(N-2)\times\mathrm{U}(1)^2$. Then, for the gauge group SU(3), the Kähler potential is explicitly computed to the two-loop order.

Keywords: Supersymmetric gauge theory, Superspaces, Supersymmetric Effective Theories, Conformal and W Symmetry.

Contents

1. Introduction

The marginal deformations [1] of $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) are a class of $\mathcal{N}=1$ superconformal field theories which have enjoyed a lot of attention in recent years. In particular, the β-deformation has been the subject of intense investigations, since its supergravity dual was found in [2]. Many aspects of the β-deformed theory have been studied at both the perturbative and nonperturbative level. In this paper we concentrate only on perturbative aspects.

An important observation of [1] is that the renormalisation group beta function vanishes (the deformation becomes exactly marginal) subject to a single, loop corrected, constraint on the deformed couplings. The nature of this constraint has been examined in both the perturbative and nonperturbative windows using a range of methods and in a variety of limits, e.g. [3 – 12] and is still a topic of ongoing discussion [13 – 17]. Despite this wealth of knowledge about the requirements for conformal invariance in β-deformed theories, the exact functional nature of the quantum corrections has received less attention [18 – 20]. The purpose of this paper is to continue in the vein of [18, 19] and investigate the structure of the two-loop Kähler potential in the β-deformed theory.

The Kähler potential is a supersymmetric generalisation of the effective potential [21] and thus it can be used to examine the renormalisation effects and vacuum structure of a quantised theory. Superfield calculations of the one-loop Kähler potential in $\mathcal{N}=1$

superspace are presented in [22 – 24], while two-loop corrections to the Wess-Zumino model have been found using superfield methods in, e.g., [25]. A computation of the two-loop Kähler potential of a general, non-renormalisable $\mathcal{N}=1$ theory was presented in [26], we will compare with this result and discuss its limitations in the conclusion. In $\mathcal{N}=1$ theories the Kähler potential is a particularly interesting sector of the low energy effective action in that it is not constrained by holomorphy in the way that the superpotential and gauge potential are. This is not the case for $\mathcal{N}=2$ theories where the non-renormalisation theorems [27, 28] imply that the Kähler potential receives only one-loop corrections and even those vanish in the case of a conformally invariant theory [28]. This means that the Kähler potential of β-deformed $\mathcal{N}=4$ SYM is purely a product of the deformation. It is for this reason that we find the Kähler potential a particularly interesting object to examine in the β-deformed SYM theory.

A major technical ingredient of any two-loop effective potential calculation is the functional form of the vacuum sunset integral. In this paper the knowledge of its structure is necessary for the presentation of the explicit conformal invariance of our results. The integral has been discussed many times in the literature, e.g. [29 – 40, 26, 19] and references therein, but to make our discussion both clearer and self-contained, we present a new, and we hope simpler, form for the two-loop integral. Like the results implied in [29 – 31], the functional form that we find is explicitly symmetric in all three masses and holds for all values of the masses, yet our result is much more compact. Our derivation is based on the method of characteristics, an approach first used in [33].

The structure of this paper is as follows: In section 2 we review the aspects of the background field quantisation of β-deformed $\mathcal{N}=4$ SYM that are necessary for our calculation, including the structure of the mass matrix for an arbitrary background in the Cartan subalgebra. Sections 3 and 4 are devoted to the calculation of the one and two-loop Kähler potentials respectively. In section 5 we make the results explicit by choosing a SU(3)-like background. Finally the appendix contains a review of the structure of the two-loop, vacuum sunset diagram.

2. Quantisation of β-deformed $\mathcal{N}=4$ SYM theory

To keep the following discussion as concise as possible, we only review the parts of the quantisation process that are necessary for the calculation of the Kähler potential, see [18, 19] for more details.

Using the superspace conventions of [41], the classical action for β-deformed $\mathcal{N}=4$ SU(N) SYM theory is

$$S = \int \mathrm{d}^8 z\, \mathrm{tr} \Phi_i^\dagger \Phi_i + \frac{1}{g^2}\int \mathrm{d}^6 z\, \mathrm{tr} W^2 + \left\{ h \int \mathrm{d}^6 z\, \mathrm{tr}(q\Phi_1\Phi_2\Phi_3 - q^{-1}\Phi_1\Phi_3\Phi_2) + \text{c.c.} \right\} ,$$

where $q = \mathrm{e}^{\mathrm{i}\pi\beta}$ is the deformation parameter, g is the gauge coupling and h is the chiral vertex coupling. The undeformed theory corresponds to the limits $q \to 1$ and $h \to g$. All of the fields transform in the adjoint representation of the gauge group and all fields are

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covariantly (anti)chiral,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\mathcal{W}_\beta = 0\,, \quad \bar{\mathcal{D}}_{\dot{\alpha}}\Phi_i = 0\,, \quad i = 1,2,3\,, \tag{2.1}$$

where $\mathcal{W}_\alpha = \mathcal{W}_\alpha^a T_a$ and $\Phi_i = \Phi_i^a T_a$ are Lie-algebra valued superfields. The generators T_a correspond to the fundamental representation of SU(N). Under the condition that the theory remains conformally invariant upon quantisation we can write g as a function of h and q, [1]. In the general case, this condition is only known to the first few orders in the loop expansion. For real β the condition for conformal invariance in the planar (large N) limit is known to be $|h| = g$ to all loops, [7, 10].

As noted in [19], it is useful to view the above $\mathcal{N}=1$ action as a pure $\mathcal{N}=2$ SYM theory (described by Φ_1 and $\mathcal{W}_\alpha$) coupled to a *deformed hypermultiplet* in the adjoint (described by $\Phi_{2,3}$). For if we quantise using only a $\mathcal{N}=2$ SYM background lying in the Cartan subalgebra then we are automatically on the Coulomb branch of the theory and all aspects of the deformation are captured in the hypermultiplet propagators and chiral cubic vertices.

Using the $\mathcal{N}=1$ background field formalism [42] we split the dynamical variables

$$\Phi_i \to \Phi_i + \varphi_i\,, \quad \mathcal{D}_\alpha \to \mathrm{e}^{-gv}\mathcal{D}_\alpha \mathrm{e}^{gv}\,, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \to \bar{\mathcal{D}}_{\dot{\alpha}}\,, \tag{2.2}$$

with lower-case letters denoting quantum superfields. To compute the quantum corrections to the Kähler potential on the Coulomb branch we choose the only non-zero background superfield to be $\Phi_1 = \Phi$ and take it to be in the Cartan subalgebra. We also need to systematically ignore all derivatives that hit the background field. Note that the absence of a background gauge field means that the covariant derivatives reduce to flat ones, i.e. $\mathcal{D}_A = D_A$. In general, a background of this type will break the gauge symmetry of the theory as $\mathrm{SU}(N) \to \mathrm{U}(1)^{N-1}$.

Gauge fixing with the supersymmetric 't Hooft gauge [43 – 47], the quadratic parts of the action are [18]

$$S^{(2)}_{\mathrm{YM}} = -1/2\int \mathrm{d}^8\mathrm{z}\,\mathrm{tr}\left(\mathrm{v}(\Box - |\mathcal{M}_{(\mathrm{g},1)}|^2)\mathrm{v}\right) + \int \mathrm{d}^8\mathrm{z}\,\mathrm{tr}\left(\varphi_1^\dagger \Box^{-1}(\Box - |\mathcal{M}_{(\mathrm{g},1)}|^2)\varphi_1\right) \tag{2.3a}$$

$$S^{(2)}_{\mathrm{hyp}} = \int \mathrm{d}^8 z\,\mathrm{tr}\left(\varphi_2^\dagger\varphi_2 + \varphi_3^\dagger\varphi_3\right) + \int \mathrm{d}^6 z\,\mathrm{tr}\varphi_3\mathcal{M}_{(h,q)}\varphi_2 + \int \mathrm{d}^6\bar{z}\,\mathrm{tr}\varphi_2^\dagger\mathcal{M}^\dagger_{(h,q)}\varphi_3^\dagger \tag{2.3b}$$

$$S^{(2)}_{\mathrm{gh}} = \int \mathrm{d}^8 z\,\mathrm{tr}\left(c^\dagger\Box^{-1}(\Box - |\mathcal{M}_{(g,1)}|^2)\tilde{c} - \tilde{c}^\dagger\Box^{-1}(\Box - |\mathcal{M}_{(g,1)}|^2)c\right)\,, \tag{2.3c}$$

where we've used the mass operators introduced in [18], which are elegantly defined by their action on a Lie-algebra valued superfield:

$$\begin{aligned}\mathcal{M}_{(h,q)}\Sigma &= h(q\Phi\Sigma - q^{-1}\Sigma\Phi) - h\frac{q-q^{-1}}{N}\mathrm{tr}(\Phi\Sigma)\mathbb{1}\\ \mathcal{M}^\dagger_{(h,q)}\Sigma &= \bar{h}(\bar{q}\Phi^\dagger\Sigma - \bar{q}^{-1}\Sigma\Phi^\dagger) - \bar{h}\frac{\bar{q}-\bar{q}^{-1}}{N}\mathrm{tr}(\Phi^\dagger\Sigma)\mathbb{1}\,.\end{aligned} \tag{2.4}$$

The relevant interactions for the two-loop diagrams of interest are the cubic couplings

$$S_{\rm I}^{(3)} = h\int \mathrm{d}^6 z\, \mathrm{tr}\left(q\varphi_1\varphi_2\varphi_3 - q^{-1}\varphi_1\varphi_3\varphi_2\right) + \mathrm{c.c.} = -\int \mathrm{d}^6 z\, (T^a_{(h,q)})^{bc}\varphi_1^a\varphi_2^b\varphi_3^c - \mathrm{c.c.} \tag{2.5a}$$

$$S_{\rm II}^{(3)} = g\int \mathrm{d}^8 z\, \mathrm{tr}\left(\varphi_i^\dagger[v,\varphi_i]\right) = -\int \mathrm{d}^8 z\, (T^a_{(g,1)})^{bc}\bar\varphi_i^a v^b\varphi_i^c\,, \tag{2.5b}$$

where, following [19], we introduce the deformed adjoint generators

$$(T^a_{(h,q)})^{bc} = -h\mathrm{tr}(qT^aT^bT^c - q^{-1}T^aT^cT^b)\,, \tag{2.6}$$

which enjoy the algebraic properties

$$(T^a_{(h,q)})^{\rm T} = T^a_{(h,-q^{-1})}\,, \quad (T^a_{(h,q)})^\dagger = T^a_{(\bar h,\bar q)}\,. \tag{2.7}$$

Note that the deformed generators can also be used to give the mass operator the compact representation $(\mathcal{M}_{(h,q)})^{ab} = \Phi^c(T^c_{(h,q^{-1})})^{ab}$.

The propagators for the action eq. (2.3) that are used in the two-loop calculation below are

$$\mathrm{i}\big\langle v(z)v^{\rm T}(z')\big\rangle = -G_{(g,1)}(z,z') \qquad \mathrm{i}\big\langle \varphi_1(z)\varphi_1^\dagger(z')\big\rangle = \frac{\bar D^2D^2}{16}G_{(g,1)}(z,z') \tag{2.8}$$

$$\mathrm{i}\big\langle \varphi_2(z)\varphi_2^\dagger(z')\big\rangle = \frac{\bar D^2D^2}{16}\overleftarrow{G}_{(h,q)}(z,z') \qquad \mathrm{i}\big\langle \bar\varphi_3(z)\varphi_3^{\rm T}(z')\big\rangle = \frac{D^2\bar D^2}{16}\overrightarrow{G}_{(h,q)}(z,z')\,, \tag{2.9}$$

where all of the fields are treated as adjoint column-vectors, in contrast to the Lie-algebraic notation used in defining the action. The Green's functions are defined by

$$\left(\Box - \mathcal{M}^\dagger_{(h,q)}\mathcal{M}_{(h,q)}\right)\overrightarrow{G}_{(h,q)}(z,z') = -\delta^8(z,z') \tag{2.10}$$

$$\left(\Box - \mathcal{M}_{(h,q)}\mathcal{M}^\dagger_{(h,q)}\right)\overleftarrow{G}_{(h,q)}(z,z') = -\delta^8(z,z')\,, \tag{2.11}$$

with the usual, causal boundary conditions. As we only have flat derivatives, the above equations are most simply solved by moving to momentum space. In the limit of vanishing deformation the mass matrices commute so that the left and right Green's functions coincide: $\overrightarrow{G}_{(g,1)} = \overleftarrow{G}_{(g,1)} = G_{(g,1)}$.

Throughout this paper we will use dimensional reduction [48] and since we only go to two loops we do not worry about any possible inconsistencies [49, 50]. This is merely a convenience, as none of the results in this paper rely on the choice of regularisation scheme and can in fact be argued at the level of the integrands.

2.1 Cartan-Weyl basis and the mass operator

The properties of the mass matrices defined in eq. (2.4) play a central role in our computations. For explicit calculations a convenient choice of basis for our gauge group is the Cartan-Weyl basis, see e.g. [51]. In this subsection we introduce some notation and a few results that will be used subsequently.

Any element in su(N) can be expanded in the Cartan-Weyl basis,

$$\psi = \psi^a T_a = \psi^{ij}E_{ij} + \psi^I H_I\,, \quad i \neq j\,, \tag{2.12}$$

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where T_a is the arbitrary basis used above and we choose our Cartan-Weyl basis as the set

$$E_{ij}\,, \quad i \neq j = 1, \ldots, N\,, \qquad H_I\,, \quad I = 1, \ldots, N-1\;. \tag{2.13}$$

The Cartan-Weyl basis satisfies[1]

$$\mathrm{tr} E_{ij} E_{kl} = \delta_{il}\delta_{jk}\,, \quad \mathrm{tr} H_I H_J = \delta_{IJ} \quad \text{and} \quad \mathrm{tr} E_{ij} H_K = 0 \tag{2.14}$$

and its elements, defined as matrices in the fundamental representation, are

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}\,, \qquad H_I = \frac{1}{\sqrt{I(I+1)}} \sum_{i=1}^{I+1} \left(1 - i\delta_{i(I+1)}\right) E_{ii}\;. \tag{2.15}$$

Since the background is chosen to be in the Cartan subalgebra,

$$\Phi = \phi^I H_I := \Phi^i E_{ii}\,,$$

the mass matrix is block diagonal when written in the Cartan-Weyl basis

$$\mathcal{M}_{(h,q)} = \begin{pmatrix} \mathcal{M}^{ijkl}_{(h,q)} & 0 \\ 0 & \mathcal{M}^{IJ}_{(h,q)} \end{pmatrix} = \begin{pmatrix} m^{ki}\delta_{il}\delta_{jk} & 0 \\ 0 & \mathcal{M}^{IJ}_{(h,q)} \end{pmatrix} \quad (\text{no sum})\,, \tag{2.16}$$

where the masses m^{ij} are defined by

$$m^{ij} = h(q\Phi^i - q^{-1}\Phi^j)\;. \tag{2.17}$$

The mass matrix in the Cartan subalgebra is symmetric, but in general not diagonal, we find

$$\mathcal{M}^{IJ}_{(h,q)} = h(q - q^{-1})\phi^K \mathrm{tr}\left(H^I H^J H^K\right) = \mathcal{M}^{JI}_{(h,q)}\;. \tag{2.18}$$

In the limit of vanishing deformation the above expression is obviously zero, and we will denote the limit of the masses in eq. (2.17) by

$$m^{ij} \xrightarrow[h=g]{q=1} m_0^{ij} = g(\Phi^i - \Phi^j)\;. \tag{2.19}$$

It is now straightforward to calculate the mass squared matrix, it is also block diagonal and has the non-zero components

$$\left(\mathcal{M}^\dagger_{(h,q)}\mathcal{M}_{(h,q)}\right)^{ijkl} = \left(\mathcal{M}_{(h,q)}\mathcal{M}^\dagger_{(h,q)}\right)^{ijkl} = |m^{ki}|^2\delta_{il}\delta_{jk} \tag{2.20a}$$

$$\left(\mathcal{M}^\dagger_{(h,q)}\mathcal{M}_{(h,q)}\right)^{IJ} = \left(\mathcal{M}_{(h,q)}\mathcal{M}^\dagger_{(h,q)}\right)^{JI} = |h(q - q^{-1})|^2\bar{\phi}^L\phi^M\eta^{IJLM}\,, \tag{2.20b}$$

where

$$\eta^{IJLM} = \mathrm{tr}\left(H^I H^J H^L H^M\right) - \frac{1}{N}\delta^{IL}\delta^{JM}\;. \tag{2.21}$$

[1]Due to our choice of normalisation the Cartan metric is just the Kronecker delta, thus we can raise and lower the group indices with impunity.

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To proceed in the one and two-loop calculations below, we will need to assume that the eigenvalues and eigenvectors of the mass squared matrix are known, that is, we know a unitary matrix U such that

$$\left(U^\dagger \mathcal{M}^\dagger_{(h,q)} \mathcal{M}_{(h,q)} U\right)^{IJ} = |m_I|^2 \delta^{IJ} \quad \text{(no sum)} \ . \tag{2.22}$$

We also need the trace of the mass squared operator. This requires the trace of η^{IJLM}, which can be found using the completeness relation for the Cartan subalgebra. The final expression is simplified by using the tracelessness of Φ to get

$$\begin{aligned} \mathrm{tr}|\mathcal{M}_{(h,q)}|^2 &= |h|^2 \left(\sum_{i\neq j} |m^{ij}|^2 + \sum_I |m_I|^2 \right) \\ &= N|h|^2 \left(|q|^2 + |q^{-1}|^2 - \frac{2}{N^2}|q-q^{-1}|^2 \right) \mathrm{tr}\Phi^\dagger\Phi \ . \end{aligned} \tag{2.23}$$

3. One-loop Kähler potential

From the quadratic terms defined in eq. (2.3) one can read off, see e.g. [52], the one-loop effective action as

$$\Gamma^{(1)} = \mathrm{iTr}\ln((\Box - |\mathcal{M}_{(h,q)}|^2)P_+) - \mathrm{iTr}\ln((\Box - |\mathcal{M}_{(g,1)}|^2)P_+) \, , \tag{3.1}$$

where Tr is both a matrix trace and a trace over full superspace, and $P_+ = (16\Box)^{-1}\bar{D}^2 D^2$ is the (flat) chiral projection operator. The matrix trace can be converted to a sum of eigenvalues using the results of section 2.1. The functional trace reduces to calculating the standard momentum integral (in $d = 4 - 2\varepsilon$ dimensions)

$$\mathcal{J}(m^2) = -\mathrm{i}\int \frac{\mu^{4-d}\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\log\left(1+\frac{m^2}{k^2}\right) = \frac{m^2}{(4\pi)^2}\left(\kappa_M - \log\frac{m^2}{M^2}\right) \, , \tag{3.2}$$

where M is an arbitrary mass scale, and $\kappa_M = \frac{1}{\varepsilon} + 2 - \log\frac{M^2}{\bar{\mu}^2} + \mathrm{O}(\varepsilon)$ with the $\overline{MS}$ renormalisation point defined by $\bar{\mu}^2 = 4\pi\mu^2 \mathrm{e}^{-\gamma}$. The function κ_M contains all of the information about the method of regularisation, e.g. if we had regularised by using a momentum cut-off at Λ^2 then we would have had $\kappa_M = 1 - \log\frac{M^2}{\Lambda^2}$.

Factoring out the integral over full superspace we get the one-loop Kähler potential

$$K^{(1)} = \sum_I \mathcal{J}(|m_I|^2) + \sum_{i\neq j}\left(\mathcal{J}(|m^{ij}|^2) - \mathcal{J}(|m_0^{ij}|^2)\right) \tag{3.3}$$

As a check on the above result, we note that it is zero in the limit of vanishing deformation. The κ_M-dependent terms are proportional to the trace of the difference of the deformed and undeformed mass matrix

$$\sum_I |m_I|^2 + \sum_{i\neq j}\left(|m^{ij}|^2 - |m_0^{ij}|^2\right) = \mathrm{tr}\left(|\mathcal{M}_{(h,q)}|^2 - |\mathcal{M}_{(g,1)}|^2\right) \ . \tag{3.4}$$

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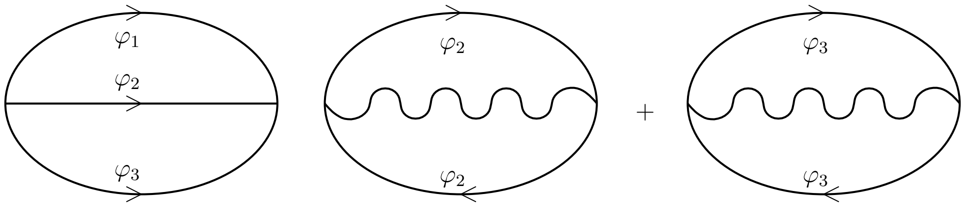

Figure 1: The two-loop diagrams contributing to the Kähler potential, Γ_1 and Γ_2 respectively. The arrows show the flow of chirality around the loop, while the fields label the propagators. The squiggly line corresponds to the $\mathcal{N}=1$ gauge superfield.

So using the trace formula eq. (2.23) it is easily seen that the above term is zero if the well known one-loop finiteness condition holds [3−5],

$$2g^2 = |h|^2\left(|q|^2 + |q^{-1}|^2 - \frac{2}{N^2}|q-q^{-1}|^2\right) \equiv 2f_q|h|^2 \ . \tag{3.5}$$

If we enforce the finiteness condition and choose M^2 to be any field generated mass then we get the explicitly super-conformal result

$$(4\pi)^2K^{(1)} = \sum_{i\neq j}\left(|m_0^{ij}|^2\log\frac{|m_0^{ij}|^2}{M^2} - |m^{ij}|^2\log\frac{|m^{ij}|^2}{M^2}\right) - \sum_I |m_I|^2\log\frac{|m_I|^2}{M^2} \ . \tag{3.6}$$

We emphasise that this result is independent of the choice of M^2.

4. Two-loop Kähler potential

In the β-deformed theory there are only four two-loop diagrams that differ from the undeformed theory [19], but some simple D-algebra shows that only two give non-zero contributions to the Kähler potential (see figure 1). Both are of the sunset type and have the generic group theoretic structure

$$\Gamma = \kappa\int \mathrm{d}^8z\,\mathrm{d}^8z' G^{ab}\mathrm{tr}\left(T^a_{(h,q^{-1})}\hat{G}_{(h,q)}T^b_{(\bar{h},\bar{q}^{-1})}\check{G}'_{(h,q)}\right) , \tag{4.1}$$

where G is an undeformed Green's function and, in general, $\hat{G}$ and $\check{G}$ denote spinor derivatives of deformed Green's functions. This decomposes in the Cartan-Weyl basis into three terms,

$$\begin{aligned}\Gamma =&\kappa|h|^2\int \mathrm{d}^8z\,\mathrm{d}^8z'\Big(G^{ijji}\big(q\bar{q}\hat{G}^{jkkj}_{(h,q)}\check{G}'^{ikki}_{(h,q)} + (q\bar{q})^{-1}\hat{G}^{kiik}_{(h,q)}\check{G}'^{kjjk}_{(h,q)}\big)\\ &+ (q(H_K)_{jj} - q^{-1}(H_K)_{ii})(\bar{q}(H_L)_{jj} - \bar{q}^{-1}(H_L)_{ii})\times\\ &\qquad\times\big(G^{KL}\hat{G}^{ijji}_{(h,q)}\check{G}'^{ijji}_{(h,q)} + G^{jiij}\hat{G}^{KL}_{(h,q)}\check{G}'^{jiij}_{(h,q)} + G^{ijji}\hat{G}^{jiij}_{(h,q)}\check{G}'^{LK}_{(h,q)}\big)\\ &+ |q-q^{-1}|^2G^{IJ}\hat{G}^{MN}_{(h,q)}\check{G}'^{LK}_{(h,q)}\mathrm{tr}(H_IH_KH_M)\mathrm{tr}(H_JH_LH_N)\Big)\\ =&\Gamma_A + \Gamma_B + \Gamma_C \ .\end{aligned} \tag{4.2}$$

We should note that if the vertices are undeformed, ie $T^a_{(h,q)} \to T^a_{(g,1)} = gT^a_{\rm ad}$, then the final term, Γ_C, is zero.

For an arbitrary background in the Cartan subalgebra Γ_A is easy to evaluate as all of its Green's functions are diagonal. To evaluate the other terms, which involve sums over the Cartan subalgebra, we will use the unitary matrices defined in eq. (2.22) to diagonalise the Green's functions,

$$(H_I)_{jj}\overleftarrow{G}^{IJ}_{(h,q)}(H_J)_{ii} = (H_I)_{ii}\overrightarrow{G}^{IJ}_{(h,q)}(H_J)_{jj} = (\bar{\mathbf{H}}_K)_{ii}G^{(K)}_{(h,q)}(\mathbf{H}_K)_{jj} \ . \tag{4.3}$$

The modified diagonal generators are defined by

$$\mathbf{H}_I = {U_I}^J H_J \,, \quad \bar{\mathbf{H}}_I = H_J {(U^\dagger)^J}_I \ . \tag{4.4}$$

In the next subsection, these generators will be combined into coefficients for the scalar loop integrals. Alternatively we could reabsorb the diagonalising unitary matrices back into the loop integrals to get a matrix valued expression. Although this does make some expressions look a bit neater and keep all of the field dependence in the now matrix valued loop integrals, to evaluate the these expressions we would still have to diagonalise the mass matrices.

4.1 Evaluation of $\Gamma_{\rm I}$

The first diagram we evaluate has the analytic expression

$$\Gamma_{\rm I} = -\frac{1}{2^8}\int \mathrm{d}^8 z\, \mathrm{d}^8 z' G^{ab}_{(g,1)}(z,z')\mathrm{tr}\Big(T^a_{(h,q^{-1})}\bar{D}^2 D^2 \overleftrightarrow{G}_{(h,q)}(z,z')T^b_{(\bar{h},\bar{q}^{-1})}D'^2\bar{D}'^2\overleftrightarrow{G}_{(h,q)}(z',z)\Big) \ . \tag{4.5}$$

For a nonzero result to occur when integrating over $\mathrm{d}^4\theta'$, all chiral derivatives have to hit the Grassmann delta functions contained in the deformed propagators. Then, writing $\mathcal{G}$ for the remaining bosonic parts of the propagators, shifting the x' integration variable to $\rho = x - x'$ and using eq. (4.2) we obtain

$$\begin{aligned} K_{\rm I} =& -|h|^2 \int \mathrm{d}^4\rho \Big\{ \mathcal{G}^{ijji}_{(g,1)}\big(q\bar{q}\mathcal{G}^{jkkj}_{(h,q)}\mathcal{G}'^{ikki}_{(h,q)} + (q\bar{q})^{-1}\mathcal{G}^{kiik}_{(h,q)}\mathcal{G}'^{kjjk}_{(h,q)}\big) \\ &+ (q(H_K)_{jj} - q^{-1}(H_K)_{ii})(\bar{q}(H_L)_{jj} - \bar{q}^{-1}(H_L)_{ii}) \times \\ &\qquad\qquad \times \big(\mathcal{G}^{KL}_{(g,1)}\mathcal{G}^{ijji}_{(h,q)}\mathcal{G}'^{ijji}_{(h,q)} + \mathcal{G}^{jiij}_{(g,1)}\overleftrightarrow{\mathcal{G}}^{KL}_{(h,q)}\mathcal{G}'^{jiij}_{(h,q)} + \mathcal{G}^{ijji}_{(g,1)}\mathcal{G}^{jiij}_{(h,q)}\overleftrightarrow{\mathcal{G}}'^{KL}_{(h,q)}\big) \\ &+ |q - q^{-1}|^2\mathcal{G}^{IJ}_{(g,1)}\overleftrightarrow{\mathcal{G}}^{MN}_{(h,q)}\overleftrightarrow{\mathcal{G}}'^{LK}_{(h,q)}\mathrm{tr}(H_I H_K H_M)\mathrm{tr}(H_J H_L H_N)\Big\} \\ =& K_{\rm IA} + K_{\rm IB} + K_{\rm IC} \ . \end{aligned} \tag{4.6}$$

By using the symmetries of the propagators, we already made some simplifications in the above expression.

Now, as all of the propagators in $K_{\rm IA}$ are already diagonal, we can move straight to momentum space and perform the ρ integral to get

$$\begin{aligned} K_{\rm IA} = -|h|^2 \sum_{i\neq j\neq k}\int \frac{\mathrm{d}^d k \mathrm{d}^d p}{(2\pi)^2 d}\frac{1}{k^2 + |m_0^{ij}|^2}\Bigg(&|q|^2\frac{1}{p^2 + |m^{kj}|^2}\frac{1}{(k+p)^2 + |m^{ki}|^2} \\ &+ |q^{-1}|^2\frac{1}{p^2 + |m^{ik}|^2}\frac{1}{(k+p)^2 + |m^{jk}|^2}\Bigg) \ . \end{aligned}$$

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Then, using the results and notation of the appendix we have

$$
K_{\mathrm{I}A} = |h|^2 \sum_{i \neq j \neq k} \Big(|q|^2 I(|m_0^{ij}|^2, |m^{ki}|^2, |m^{kj}|^2) + |q^{-1}|^2 I(|m_0^{ij}|^2, |m^{ik}|^2, |m^{jk}|^2) \Big) \ . \tag{4.7}
$$

To evaluate $K_{\mathrm{I}B}$ we diagonalise the propagators, as described above. The result is

$$
\begin{aligned}
K_{\mathrm{I}B} = &- |h|^2 \sum_{i \neq j, K} \int \mathrm{d}^4\rho \, (q(\bar{\mathbf{H}}_K)_{jj} - q^{-1}(\bar{\mathbf{H}}_K)_{ii})(\bar{q}(\mathbf{H}_K)_{jj} - \bar{q}^{-1}(\mathbf{H}_K)_{ii}) \times \\
&\qquad \times \Big(\mathcal{G}^{(K)}_{(g,1)} \mathcal{G}^{ijji}_{(h,q)} \mathcal{G}'^{ijji}_{(h,q)} + \mathcal{G}^{jiij}_{(g,1)} \mathcal{G}^{(K)}_{(h,q)} \mathcal{G}'^{jiij}_{(h,q)} + \mathcal{G}^{ijji}_{(g,1)} \mathcal{G}^{jiij}_{(h,q)} \mathcal{G}'^{(K)}_{(h,q)} \Big) \qquad (4.8) \\
= &|h|^2 \sum_{i \neq j, K} \varpi_{\bar{q},Kij} \Big(\, I(\,0\,, |m^{ji}|^2, |m^{ji}|^2) + 2I(|m_0^{ij}|^2, |m^{ij}|^2, |m_K|^2) \Big) \ ,
\end{aligned}
$$

where $\varpi_{q,Kij}$ is defined by

$$
\varpi_{q,Kij} = \big(q(\mathbf{H}_K)_{ii} - q^{-1}(\mathbf{H}_K)_{jj} \big) \big(\bar{q}(\bar{\mathbf{H}}_K)_{ii} - \bar{q}^{-1}(\bar{\mathbf{H}}_K)_{jj} \big) \ , \quad \text{(no sum)} \ . \tag{4.9}
$$

Similarly we evaluate $K_{\mathrm{I}C}$ to find

$$
K_{\mathrm{I}C} = |h(q - q^{-1})|^2 \sum_{I,J} \boldsymbol{\eta}_{IJ} I(0, |m_I|^2, |m_J|^2) \, , \tag{4.10}
$$

where $\boldsymbol{\eta}$ is closely related to η, defined in eq. (2.21),

$$
\boldsymbol{\eta}_{IJ} = \mathrm{tr}(\bar{\mathbf{H}}_I \bar{\mathbf{H}}_J \mathbf{H}_I \mathbf{H}_J) - \frac{1}{N} (U^T U)_{IJ} (U^\dagger U^*)_{JI} \, , \quad \text{(no sum)} \ . \tag{4.11}
$$

Note that in general the coefficients $\varpi_{q,Kij}$ and $\boldsymbol{\eta}_{IJ}$ are functions of ratios of the background dependent masses.

4.2 Evaluation of Γ_{II}

The second diagram,

$$
\begin{aligned}
\Gamma_{\mathrm{II}} = \frac{1}{2^9} \int \mathrm{d}^8 z \, \mathrm{d}^8 z' G^{ab}_{(g,1)}(z, z') \mathrm{tr} \Big(&T^a_{(g,1)} \bar{D}^2 D^2 \overset{\hookleftarrow}{G}_{(h,q)}(z, z') T^b_{(g,1)} D'^2 \bar{D}'^2 \overset{\hookleftarrow}{G}_{(h,q)}(z', z) \\
&+ T^a_{(g,1)} \bar{D}^2 D^2 \overset{\hookrightarrow}{G}_{(h,q)}(z, z') T^b_{(g,1)} D'^2 \bar{D}'^2 \overset{\hookrightarrow}{G}_{(h,q)}(z', z) \Big) \, ,
\end{aligned} \tag{4.12}
$$

is simpler due to the lack of deformed vertices. Following the same procedure as above we find $K_{\mathrm{II}} = K_{\mathrm{II}A} + K_{\mathrm{II}B}$, with

$$
K_{\mathrm{II}A} = -g^2 \sum_{i \neq j \neq k} \Big(\, I(|m_0^{ji}|^2, |m^{kj}|^2, |m^{ki}|^2) + I(|m_0^{ji}|^2, |m^{ik}|^2, |m^{jk}|^2) \, \Big) \tag{4.13}
$$

and

$$
K_{\mathrm{II}B} = -g^2 \sum_{i \neq j, K} \varpi_{1,Kij} \Big(\, I(\,0\,, |m^{ji}|^2, |m^{ji}|^2) + 2I(|m_0^{ij}|^2, |m^{ij}|^2), |m_K|^2 \Big) \ . \tag{4.14}
$$

4.3 Finiteness and conformal invariance

Combining the two diagrams we see that, like the one-loop, the two-loop Kähler potential is written as the difference of terms that cancel in the limit of vanishing deformation:

$$\begin{aligned} K^{(2)} = &\sum_{i\neq j\neq k} \left((|hq|^2 - g^2)\, I(|m_0^{ij}|^2, |m^{ki}|^2, |m^{kj}|^2) + (|hq^{-1}|^2 - g^2) I(|m_0^{ij}|^2, |m^{ik}|^2, |m^{jk}|^2) \right) \\ &+ \sum_{i\neq j, K} \left(|h|^2 \varpi_{\bar{q},Kij} - g^2 \varpi_{1,Kij} \right) \Big(I(0, |m^{ji}|^2, |m^{ji}|^2) + 2I(|m_0^{ij}|^2, |m^{ij}|^2, |m_K|^2) \Big) \\ &+ |h(q - q^{-1})|^2 \sum_{I,J} \boldsymbol{\eta}_{IJ} I(0, |m_I|^2, |m_J|^2) \ . \end{aligned} \tag{4.15}$$

As described in the appendix, the two-loop integral, $I(x, y, z)$, can be decomposed as

$$I(x, y, z) = \iota(x) + \iota(y) + \iota(z) + \mathbb{I}(x, y, z) \,, \quad \mathbb{I}(x, y, z) = -\tilde{\xi}(x, y, z)/2 \,, \tag{4.16}$$

where the ι terms include all of the divergences and renormalisation point dependence, and $\tilde{\xi}$, defined in eq. (A.18), is a function of mass ratios only. Since the masses are disentangled in the ι terms, the sums can be simplified by using the following identities:[2]

$$\sum_K \varpi_{q,Kij} = \frac{1}{N} \sum_{i\neq j} \varpi_{q,Kij} = |q|^2 + \frac{1}{|q|^2} - \frac{1}{N} |q - q^{-1}|^2 := 2g_q \tag{4.17a}$$

$$\sum_J \boldsymbol{\eta}_{IJ} = \sum_J \eta_{IIJJ} = \frac{N-2}{N} \ . \tag{4.17b}$$

The result is that all ι dependence can be collected into

$$\begin{aligned} K_\iota^{(2)} =& N \left(|h|^2 \left(|q|^2 + |q^{-1}|^2 - \frac{2}{N^2} |q - q^{-1}|^2 \right) - 2g^2 \right) \\ &\times \left(\sum_{i\neq j} \Big(\iota(|m_0^{ij}|^2) + 2\iota(|m^{ij}|^2) \Big) + 2 \sum_I \iota(|m_I|^2) \right) \ . \end{aligned} \tag{4.18}$$

From the above expression and the trace formulae given in section 2.1 we may read off the quadratic dependence of the Kähler potential:

$$K_{\text{quad}}^{(2)} \propto 4N^2 (|h|^2 f_q - g^2)(2|h|^2 f_q + g^2) \text{tr} \Phi^\dagger \Phi \,,$$

where the constant of proportionality is a number that is subtraction scheme dependent and f_q is the function that occurs in the one-loop finiteness condition eq. (3.5). The above prefactor is, for good reason, reminiscent of the general expression for the two-loop anomalous dimension given in, for example, [53 – 56].

So, as expected, the two-loop Kähler potential is finite and independent of the renormalisation point if the one-loop finiteness condition, eq. (3.5), is satisfied. It is interesting to note that the 'meaning' of eq. (3.5) is different at one and two-loops. At one-loop it

[2] Note that using eq. (4.17a) it becomes possible to perform the sum over K in the first term of the middle line of eq. (4.15).

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implies that the trace of the mass matrix is invariant under the deformation, while at two loops it implies that the coefficients of the scalar diagrams sum to zero.

If we enforce the finiteness condition, eq. (3.5), then we get the explicitly superconformal two-loop Kähler potential by making the replacements $g^2 \to |h|^2 f_q$ and $I \to \mathbb{I}$ in eq. (4.15).

5. Special backgrounds

In the above analysis the background superfield pointed in an arbitrary direction in the Cartan subalgebra of $\mathrm{SU}(N)$. In order to make our previous analysis concrete we now choose the specific background

$$\Phi = \sqrt{N(N-1)}\phi_1 H_{N-1} + \sqrt{(N-1)(N-2)}\phi_2 H_{N-2} \ . \tag{5.1}$$

The characteristic feature of this background is that it leaves the subgroup $\mathrm{U}(1)^2 \times \mathrm{SU}(N-2)$ of $\mathrm{SU}(N)$ unbroken. The two $\mathrm{U}(1)$s are associated with the generators H_{N-1} and H_{N-2}. In the limit $\phi_2 \to 0$, we obtain the background previously used for the calculation of the two-loop Kähler potential in [19].

There are twelve different, nonzero masses that occur with this background. There are nine deformed masses:

$$\begin{aligned}
&m_1^2 = |m^{ij}|^2 = |m_I|^2 = |h(q-q^{-1})|^2 |\phi_1+\phi_2|^2 \,, \\
&m_2^2 = |m^{i(N-1)}|^2 = |h(q-q^{-1})\phi_1 + h(q+(N-2)q^{-1})\phi_2|^2 \,, \quad && m_{\tilde{2}}^2 = |m^{(N-1)j}|^2 = m_2^2\big|_{q\to q^{-1}} \,, \\
&m_3^2 = |m^{iN}|^2 = |hq\phi_2 + h(q+(N-1)q^{-1})\phi_1|^2 \,, && m_{\tilde{3}}^2 = |m^{Nj}|^2 = m_3^2\big|_{q\to q^{-1}} \,, \\
&m_4^2 = |m^{(N-1)N}|^2 = |h(N-2)q\phi_2 - h(q+(N-1)q^{-1})\phi_1|^2 \,, && m_{\tilde{4}}^2 = |m^{N(N-1)}|^2 = m_4^2\big|_{q\to q^{-1}} \,, \\
&m_\pm^2 = 1/2\, |\mathrm{h}(\mathrm{q}-\mathrm{q}^{-1})|^2 \left(\mathrm{a}+\mathrm{c} \pm \sqrt{(\mathrm{a}-\mathrm{c})^2 + 4|\mathrm{b}|^2}\right) \,,
\end{aligned} \tag{5.2a}$$

where the indices i, j and I range from 1 to $(N-2)$ and $(N-3)$ respectively, and their three undeformed counterparts:

$$\begin{aligned}
m_{02}^2 &= |m_0^{i(N-1)}|^2 = |m_0^{(N-1)j}|^2 = g^2(N-1)^2|\phi_2|^2 \,, \\
m_{03}^2 &= |m_0^{iN}|^2 = |m_0^{Nj}|^2 = g^2|N\phi_1+\phi_2|^2 \,, \\
m_{04}^2 &= |m_0^{(N-1)N}|^2 = |m_0^{N(N-1)}|^2 = g^2|N\phi_1-(N-2)\phi_2|^2 \ .
\end{aligned} \tag{5.2b}$$

The quantities a, b and c come from the Cartan subalgebra block of the mass matrix, which is diagonal except for the bottom 2×2 block:

$$(\mathcal{M}^\dagger_{(h,q)}\mathcal{M}_{(h,q)})_{IJ} = |h(q-q^{-1})|^2 \left(\begin{array}{ccc:cc}
|\phi_1+\phi_2|^2 & & & & \\
& \ddots & & & \\
& & |\phi_1+\phi_2|^2 & & \\
\hdashline
& & & a & b \\
& & & b^* & c
\end{array}\right) , \tag{5.3a}$$

$$a = \left((N-3)^2 + 1 - 2/N\right)\bar\phi_2\phi_2 + \bar\phi_1\phi_1 - (N-3)(\bar\phi_2\phi_1 + \bar\phi_1\phi_2) \tag{5.3b}$$

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$$
\begin{aligned}
b &= \sqrt{1-2/N}\,\big((3-N)\bar\phi_2\phi_2+\bar\phi_1\phi_2-(N-2)\bar\phi_2\phi_1\big)\;, && (5.3c)\\
c &= (N-2)^2\bar\phi_1\phi_1+\big(1-2/N\big)\bar\phi_2\phi_2\;. && (5.3d)
\end{aligned}
$$

The eigenvalues are m_1^2 and $m_\pm^2$ with the corresponding orthonormal eigenvectors

$$
e_{I<N-2}\,, \quad v_\pm=\frac{(0,\ldots,0,a-c\pm\sigma,2b^*)}{\sqrt{2\sigma(\sigma\pm(a-c))}}\,, \quad \sigma=\sqrt{(a-c)^2+4|b|^2} \tag{5.4}
$$

where e_I is the standard basis vector with a one in the I^{th} position and zero everywhere else. Note that eq. (5.3a) is diagonal when $\phi_2=0$ (including the SU(2) case) and in the planar limit, when $N\to\infty$.

The one-loop Kähler potential is simply read from eq. (3.3):

$$
\begin{aligned}
K^{(1)} =&\mathcal{J}(m_+^2)+\mathcal{J}(m_-^2)-2(N-2)\big(\mathcal{J}(m_{02}^2)+\mathcal{J}(m_{03}^2)\big)-2\mathcal{J}(m_{04}^2)+(N^2-2N-1)\mathcal{J}(m_1^2)\\
&+(N-2)\big(\mathcal{J}(m_2^2)+\mathcal{J}(m_{\tilde 2}^2)+\mathcal{J}(m_3^2)+\mathcal{J}(m_{\tilde 3}^2)\big)+\mathcal{J}(m_4^2)+\mathcal{J}(m_{\tilde 4}^2)\;.
\end{aligned}\tag{5.5}
$$

The effect of enforcing the finiteness condition is to replace $(4\pi)^2\mathcal{J}(x)$ by $x\log(M^2/x)$ for an arbitrary field dependent mass term M^2. Similarly, the two-loop Kähler potential is read from eq. (4.15):

$$
\begin{aligned}
K^{(2)} =&(N-2)\big(|hq|^2-g^2\big)\Big[(N-3)(N-4)I(0,1,1)+2I(2_0,\tilde 3,\tilde 4)+2I(3_0,\tilde 2,4)+2I(4_0,2,3)\\
&\qquad+(N-3)\big(2I(2_0,2,1)+2I(3_0,3,1)+I(0,\tilde 2,\tilde 2)+I(0,\tilde 3,\tilde 3)\big)\Big]+\big[q\to q^{-1}\big]\\
&+(|h|^2g_q-g^2)\Big[(N-2)\Big((N-3)I(0,1,1)\\
&\qquad+2I(0,2,2)+2I(0,3,3)\Big)+2I(0,4,4)\Big]+\big[q\to q^{-1}\big]\\
&+2\sum_{i\neq j}\Big[\big(2|h|^2g_q-2g^2-\varpi'_{\bar q,2ij}-\varpi'_{\bar q,1ij}\big)I(|m_0^{ij}|^2,|m^{ij}|^2,1)\\
&\qquad+\varpi'_{\bar q,2ij}I(|m_0^{ij}|^2,|m^{ij}|^2,+)+\varpi'_{\bar q,1ij}I(|m_0^{ij}|^2,|m^{ij}|^2,-)\Big]\\
&+|h(q-q^{-1})|^2\Big[\big((1-2/N)(N-5)+\boldsymbol{\eta}'_{22}+2\boldsymbol{\eta}'_{21}+\boldsymbol{\eta}'_{11}\big)I(0,1,1)\\
&\qquad+2(1-2/N-\boldsymbol{\eta}'_{22}-\boldsymbol{\eta}'_{21})I(0,+,1)+2(1-2/N-\boldsymbol{\eta}'_{12}-\boldsymbol{\eta}'_{11})I(0,-,1)\\
&\qquad+\boldsymbol{\eta}'_{22}I(0,+,+)+2\boldsymbol{\eta}'_{21}I(0,-,+)+\boldsymbol{\eta}'_{11}I(0,-,-)\Big],
\end{aligned}\tag{5.6}
$$

where we have introduced a condensed notation for the masses

$$
m_\pm^2\sim\pm\,,\quad m_i^2\sim i\quad\text{and}\quad m_{0i}^2\sim i_0
$$

with $i=1,\tilde 1,\ldots,4,\tilde 4$ and defined

$$
\varpi'_{q,Kij}=|h|^2\varpi_{q,(N-K)ij}-g^2\varpi_{1,(N-K)ij}\,,\quad \boldsymbol{\eta}'_{IJ}=\boldsymbol{\eta}_{(N-I)(N-J)}\;.
$$

We've also used eq. (4.17) to make the expression only dependent on $\varpi'_{q,Iij}$ and $\boldsymbol{\eta}'_{IJ}$ for $I,J=1,2$. The coefficients, $\varpi_{q,Kij}$ and $\boldsymbol{\eta}_{IJ}$, are then calculated using the results

$$
\begin{aligned}
&\mathbf{H}_I=H_I\,, && I<N-2\,,\\
&\mathbf{H}_{N-2}=(v_+)_{N-2}H_{N-2}+(v_-)_{N-2}H_{N-1}\,, && \mathbf{H}_{N-1}=(v_+)_{N-1}H_{N-2}+(v_-)_{N-1}H_{N-1}\;.
\end{aligned}
$$

We emphasise that $\mathbf{H}_I$ and therefore $\varpi_{q,Kij}$ and $\boldsymbol{\eta}_{IJ}$ are in general field dependent quantities.

We now examine the two limiting cases, $\phi_2 \to 0$ and $N \to 3$. In both these limits we find that the coefficients $\varpi'_{q,Iij}$ and $\boldsymbol{\eta}'_{IJ}$ are independent of the background fields, which is not representative of the general case.

In the case where $\phi_2 \to 0$ the entire mass matrix is diagonal, so that the unitary, diagonalising matrix is just the unit matrix. Thus the coefficients ϖ and $\boldsymbol{\eta}$ are background independent, and can be calculated in closed form. Also, similarly to eq. (5.6), we can write $K^{(2)}$ such that we only need to know $\boldsymbol{\eta}_{(N-1)(N-1)}$ and $\varpi_{q,(N-1)ij}$, which further eases the calculational load. If we enforce conformal invariance then the sole mass scale, $\bar{\phi}_1\phi_1$, must cancel in all of the mass ratios, so that the full, quantum corrected, Kähler potential is just a deformation dependent rescaling of the classical Kähler potential [19]. Finally, if we choose a real deformation, the limit of our two-loop result reproduces equation (6.5) of [19] exactly, which is a good check of our method.

When the gauge group is SU(3) the terms with the mass m_1^2 no longer appear in the summations, $m_\pm^2$ is compactly written as $|h(q-q^{-1})(\phi_1 \mp \frac{\mathrm{i}}{\sqrt{3}}\phi_2)|^2$ and the rest of the masses take the obvious limits. We will assume that we are on the conformal surface and set $g^2 = f_q|h|^2$. The one-loop Kähler potential does not simplify much, choosing $M^2 = f_q|h\phi|^2$ where $|\phi|^2 = \mathrm{tr}\Phi^\dagger\Phi \neq 0$, we have

$$\begin{aligned}\frac{(4\pi)^2}{|h|^2}K^{(1)}_{\mathrm{SU}(3)} =& 2f_q\left(|2\phi_2|^2\log\frac{|2\phi_2|^2}{|\phi|^2} + |3\phi_1+\phi_2|^2\log\frac{|3\phi_1+\phi_2|^2}{|\phi|^2} + (\phi_2 \to -\phi_2)\right)\\ &- |q-q^{-1}|^2\left(|\phi_1 - \frac{\mathrm{i}}{\sqrt{3}}\phi_2|\log\frac{|\phi_1 - \mathrm{i}\phi_2/\sqrt{3}\,|}{f_q|q-q^{-1}|^{-2}|\phi|^2} + (\phi_2 \to -\phi_2)\right)\\ &- \Bigg(|(q-q^{-1})\phi_1+(q+q^{-1})\phi_2|^2\log\frac{|(q-q^{-1})\phi_1+(q+q^{-1})\phi_2|^2}{f_q|\phi|^2}\\ &+ |(q+2q^{-1})\phi_1+q\phi_2|^2\log\frac{|(q+2q^{-1})\phi_1+q\phi_2|^2}{f_q|\phi|^2} + (\phi_2\to-\phi_2)\Bigg) + \left(q\to q^{-1}\right)\ .\end{aligned}$$

Although we can combine the logarithms and explicitly remove all reference to $|\phi|^2$, the analytic structure and the various limits are simpler to examine in the above form.

To find the two-loop Kähler potential, we choose the diagonalising unitary matrix to be

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix} \qquad \Longrightarrow \mathbf{H}_2 = (\mathrm{i}\mathbf{H}_1)^* = \frac{-1}{\sqrt{3}}\mathrm{diag}(r_+, r_-, -1)$$

where -1 and $r_\pm = (1\pm\mathrm{i}\sqrt{3})/2$ are the cube roots of minus one. Then it is straightforward to compute

$$\varpi_{q,Kij} = \frac{1}{3}\left[\begin{pmatrix}|q-q^{-1}|^2 & |qr_- - q^{-1}r_+|^2 & |qr_- + q^{-1}|^2\\ |qr_+ - q^{-1}r_-|^2 & |q-q^{-1}|^2 & |qr_+ + q^{-1}|^2\\ |q+q^{-1}r_-|^2 & |q+q^{-1}r_+|^2 & |q-q^{-1}|^2\end{pmatrix}, q\to q^{-1}\right]\ , \quad \boldsymbol{\eta}_{IJ} = \frac{1}{3}\delta_{IJ}\ .$$

We split the two-loop Kähler potential into $K^{(2)}_{\mathrm{SU}(3)} = K_{\mathrm{A}} + (q\to q^{-1}) + K_{\mathrm{B}} + (q \to q^{-1})$ where the labelling follows the decomposition eq. (4.2). Note that in the case being considered $K_C = 0$, since it only contributes terms of the form $\mathbb{I}(0,x,x)$ which are zero from eq. (A.18). This is also true for the integrals that come from the first terms in K_{IB} and K_{IIB}. Substituting in the masses and using the fact that $\mathbb{I}(x,y,z)$ is a homogeneous function of order one to pull out a factor of $|h|^2$, we find

$$
\begin{aligned}
K_{\rm A}^{(2)} =& 2|h|^4(|q|^2 - f_q)\Big[\mathbb{I}\big(f_q|2\phi_2|^2, |(2q+q^{-1})\phi_1 + q^{-1}\phi_2|^2, |(2q+q^{-1})\phi_1 - q^{-1}\phi_2|^2\big) \\
&+ \mathbb{I}\big(f_q|3\phi_1+\phi_2|^2, |(q-q^{-1})\phi_1 - (q+q^{-1})\phi_2|^2, |(q+2q^{-1})\phi_1 - q\phi_2|^2\big) \\
&+ \mathbb{I}\big(f_q|3\phi_1-\phi_2|^2, |(q-q^{-1})\phi_1 + (q+q^{-1})\phi_2|^2, |(q+2q^{-1})\phi_1 + q\phi_2|^2\big)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
K_{\rm B}^{(2)} =& 2|h|^4\Bigg[\left(\frac{1}{3}|qr_- - q^{-1}r_+|^2 - f_q\right) \\
&\times\left[\mathbb{I}\left(f_q|2\phi_2|^2, |(q-q^{-1})\phi_1 + (q+q^{-1})\phi_2|^2, \left|(q-q^{-1})\left(\phi_1 - \frac{\mathrm{i}}{\sqrt{3}}\phi_2\right)\right|^2\right)\right. \\
&\left.+\,\mathbb{I}\left(f_q|2\phi_2|^2, |(q-q^{-1})\phi_1 - (q+q^{-1})\phi_2|^2, \left|(q-q^{-1})\left(\phi_1 + \frac{\mathrm{i}}{\sqrt{3}}\phi_2\right)\right|^2\right)\right] \\
&+\left(\frac{1}{3}|qr_- + q^{-1}|^2 - f_q\right)\left[\mathbb{I}\left(f_q|3\phi_1+\phi_2|^2, |(q+2q^{-1})\phi_1 + q\phi_2|^2, \left|(q-q^{-1})\left(\phi_1 - \frac{\mathrm{i}}{\sqrt{3}}\phi_2\right)\right|^2\right)\right. \\
&\left.+\,\mathbb{I}\left(f_q|3\phi_1+\phi_2|^2, |(2q+q^{-1})\phi_1 + q^{-1}\phi_2|^2, \left|(q-q^{-1})\left(\phi_1 + \frac{\mathrm{i}}{\sqrt{3}}\phi_2\right)\right|^2\right)\right] \\
&+\left(\frac{1}{3}|qr_+ + q^{-1}|^2 - f_q\right)\left[\mathbb{I}\left(f_q|3\phi_1-\phi_2|^2, |(q+2q^{-1})\phi_1 - q\phi_2|^2, \left|(q-q^{-1})\left(\phi_1 - \frac{\mathrm{i}}{\sqrt{3}}\phi_2\right)\right|^2\right)\right. \\
&\left.+\,\mathbb{I}\left(f_q|3\phi_1-\phi_2|^2, |(2q+q^{-1})\phi_1 - q^{-1}\phi_2|^2, \left|(q-q^{-1})\left(\phi_1 + \frac{\mathrm{i}}{\sqrt{3}}\phi_2\right)\right|^2\right)\right]\Bigg]
\end{aligned}
$$

From the expressions for $\mathbb{I} = -\tilde{\xi}/2$ given in the appendix, eq. (A.18) and eq. (A.22), we see that the above form is scale invariant. We note that taking the deformation to be real does not provide much simplification, except when $\phi_2 = 0$ and a real deformation makes the tilded masses equal to their non-tilded counterparts.

6. Conclusion

The above calculations show that although it is conceptually straightforward to calculate the loop corrections to the Kähler potential of β-deformed $\mathcal{N}=4$ SYM on the Coulomb branch, the details of the calculation are quite involved for an arbitrary background. This is because not only do the $(3N-2)(N-1)/2$ eigenmasses enter the result, but also the field dependent eigenvectors.

To help reveal the general structure of the Kähler potential it is useful to use the idea of matrix valued loop integrals (see e.g. [26]) discussed in section 4. Then all field dependence is in the loop integrals, for example

$$\sum_{IJ}\boldsymbol{\eta}_{IJ}I(0,m_I^2,m_J^2) = \sum_{IJKL}\eta_{IJKL}I(0,(\mathcal{M}^\dagger\mathcal{M})^{IJ},(\mathcal{M}\mathcal{M}^\dagger)^{KL})\ .$$

Thus we see that, assuming the finiteness condition is enforced, the general conformally invariant structure of the Kähler potential can be written in terms of a function of the $(5N-2)(N-1)/2$ components of the mass matrix eq. (2.20)

$$K(\Phi^\dagger,\Phi) = |\phi|^2\, F\left(\frac{|g(\Phi^i-\Phi^j)|^2}{|\phi|^2}, \frac{|h(q\Phi^i-q^{-1}\Phi^j)|^2}{|\phi|^2}, \frac{|h(q-q^{-1})|^2\bar\phi^L\phi^M\eta_{IJLM}}{|\phi|^2}\right),$$

where we remember that we have chosen the background to be $\Phi = H_I\phi^I = E_{ii}\Phi^i$. For definiteness, we have inserted the nonvanishing $|\phi|^2 = \mathrm{tr}\Phi^\dagger\Phi = \sum_I|\phi^I|^2 = \sum_i|\Phi^i|^2$ into all terms in the above expression, but in general this is not necessary. The loop corrections to the Kähler potential are identically zero in the limit of vanishing deformation, thus F can always be written as one (for the tree level term) plus the difference between two terms that become identical as the deformation is switched off.

Finally, we should compare our results to the two-loop Kähler potential calculation of [26]. They examined the Kähler potential of a general, non-renormalisable $\mathcal{N}=1$ theory with the assumption that the background chiral fields satisfy the classical equations of motion. This assumption turns out to be suitable for special background configurations (such as vacuum valleys) but appears to be incompatible with the Kähler approximation for general backgrounds [57]. The point is that the derivatives of the background chiral fields are systematically ignored when computing the quantum corrections to the Kähler potential. Then, for (dynamically) massive fields, enforcing the equations of motion restricts the background fields to discrete or vanishing values.[3] (For example, a simple model where enforcing the equations of motion will lead to a vanishing Kähler potential is massive supersymmetric QED.[4] The classical equations of motion are $\phi_\pm = \frac{1}{m}\bar D^2\bar\phi_\mp$ which imply that the background fields are zero in the Kähler approximation.) With that said, there are many interesting theories with background configurations that do not have the above problem, e.g. the Coulomb branch of both $\mathcal{N}=2$ and β-deformed SYM theories. A comparison between the final results of [26] and our initial expressions (before going to the Cartan-Weyl basis) has been made and it was found that the two results match.[5]

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[3] Similar problems occur in non-supersymmetric theories when the equations of motion are assumed in an effective potential calculation.

[4] The two-loop Euler-Heisenberg Lagrangian and the one-loop Kähler potential for SQED were studied in [58]. The one-loop Kähler potential was also studied in a general, two-parameter R_ξ-gauge [59].

[5] Note that the match only occured after an error in `hep-th/0511004` was corrected.

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A. Closed form for $I(x, y, z)$

In this appendix we examine the calculation of the two-loop vacuum diagram

$$I(x,y,z) = \mu^{4\varepsilon} \int \frac{\mathrm{d}^d k \mathrm{d}^d p}{(2\pi)^{2d}} \frac{1}{(k^2+x)(p^2+y)((k+p)^2+z)} \tag{A.1}$$

where we work in a $d = 4 - 2\varepsilon$ dimensional, Euclidean space-time and x, y and z are three independent square masses. In the literature there are four main approaches to calculating this integral. It can be directly calculated, as in [34] where the Mellin-Barnes representation for the propagators is used, or it can be calculated indirectly by exploiting the different types of differential equations [60, 61] that $I(x, y, z)$ has to satisfy. The first differential equation is the homogeneity equation,

$$(1 - 2\varepsilon - x\partial_x - y\partial_y - z\partial_z) I(x, y, z) = 0\,, \tag{A.2}$$

and was used in [29−31] to express $I(x, y, z)$ in terms of its first derivatives, which have a more amenable Feynman parameterisation.[6] The second type of differential equation is the ordinary differential equation of [38].[7] The final approach is the partial differential equation used in [33], and is the approach that we'll re-examine here.

Although all approaches must yield equivalent results, only those of [38] and [34] had been analytically shown to be the same (to the authors knowledge). This appendix will take this one step further and show the equivalence of the Clausen function form for $I(x, y, z)$ given in [34] to the result of [33] which is expressend in terms of Lobachevsky functions. Along the way we find a completely symmetric representation for $I(x, y, z)$ that holds for all values of the masses.

By using the integration by parts technique [62, 63], one can see that $I(x, y, z)$ must satisfy the following differential equation

$$[(z-y)\partial_x + \text{cycl.}]\, I(x,y,z) = \left[J'(x)\big(J(y) - J(z)\big) + \text{cycl.}\right]\,, \tag{A.3}$$

where $J(x)$ is the one-loop tadpole integral

$$J(x) = \mu^{2\varepsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2+x} = \frac{\mu^{2\varepsilon}}{(4\pi)^{2-\varepsilon}} \Gamma(\varepsilon - 1) x^{1-\varepsilon}\,. \tag{A.4}$$

In [33] it was noted that eq. (A.3) can be solved using the method of characteristics. To do this we need to introduce a one-parameter flow (x_t, y_t, z_t) such that

$$\dot{x}_t = y_t - z_t\,, \quad \dot{y}_t = z_t - x_t\,, \quad \dot{z}_t = x_t - y_t\,, \tag{A.5}$$

which allows us to write eq. (A.3) as

$$\frac{\mathrm{d}}{\mathrm{d}t} I(x_t, y_t, z_t) = -\Gamma' \left(\dot{x}_t (y_t z_t)^{-\varepsilon} + \dot{y}_t (z_t x_t)^{-\varepsilon} + \dot{z}_t (x_t y_t)^{-\varepsilon}\right)\,, \tag{A.6}$$

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[6] This approach was extended to non-vacuum diagrams in [36].

[7] Also used in [32] for the case of two equal masses.

where $(4\pi)^d\Gamma' = \mu^{4\varepsilon}\Gamma(\varepsilon)\Gamma(\varepsilon-1)$. The flow eq. (A.5) has two algebraically independent invariants, we choose

$$c = x_t + y_t + z_t\,, \qquad \Delta = 2(x_t y_t + y_t z_t + z_t x_t) - x_t^2 - y_t^2 - z_t^2\,, \tag{A.7}$$

where Δ is known as the "triangle" function and is related to the negative of the Källen function. Using these invariants we can write

$$y_t z_t = \left(x_t - \frac{c}{2}\right)^2 + \frac{\Delta}{4}\,, \quad \text{and cycl.} \tag{A.8}$$

allowing us to integrate the flow equation in the form

$$I(x,y,z) = I(x_0,y_0,z_0) - \Gamma'\left(\int_{x_0-c/2}^{x-c/2} + \int_{y_0-c/2}^{y-c/2} + \int_{z_0-c/2}^{z-c/2}\right)\frac{\mathrm{d}s}{(s^2+\Delta/4)^\varepsilon}\,, \tag{A.9}$$

where the end point of the flow has been chosen as $(x_1,y_1,z_1) = (x,y,z)$. We can now choose the flow's initial point so that the integral $I(x_0,y_0,z_0)$ is more easily evaluated. In [33] the flow was chosen to start at $(X,Y,0)$, a choice that is only good for $\Delta \le 0$, while in [19] the case of $\Delta > 0$ was examined using the initial point (X,Y,Y). In this discussion we make the latter choice for all values of Δ. The masses X and Y can be seen to be real and non-negative when written in terms of the flow invariants using

$$c = x_t + y_t + z_t = X + 2Y\,, \quad \Delta = 2(x_t y_t + y_t z_t + z_t x_t) - x_t^2 - y_t^2 - z_t^2 = X(4Y - X)\,. \tag{A.10}$$

Although the explicit form of $I(X,Y,Y)$ is known, we will once again follow [33, 19] and make a second flow based on the differential equation

$$(X\partial_X + (1/2\,\mathrm{X} - \mathrm{Y})\,\partial_Y)\,I(X,Y,Y) = \Gamma'\frac{X^{1-\varepsilon} - Y^{1-\varepsilon}}{Y^\varepsilon}\,, \tag{A.11}$$

which is solved by introducing another flow

$$\dot{X}_t = X_t\,, \quad \dot{Y}_t = 1/2\,\mathrm{X_t} - \mathrm{Y_t}\,, \quad \mathrm{X_1} = \mathrm{X}\,, \quad \mathrm{Y_1} = \mathrm{Y}\,. \tag{A.12}$$

This flow also conserves the triangle function, Δ, but c is no longer preserved. We now need to choose different starting points for the flow, depending on the sign of Δ, specifically we choose

$$(X_0, Y_0) = (\sqrt{-\Delta}, 0) \quad \text{and} \quad (X_0, Y_0) = (\sqrt{\Delta/3}, \sqrt{\Delta/3}) \tag{A.13}$$

for $\Delta < 0$ and $\Delta > 0$ respectively. The differential equation may now be integrated, yielding

$$I(x,y,z) = \Gamma'\left[G\left(\frac{c}{2} - x\right) + \text{cycl.}\right] + \begin{cases} I\left(\sqrt{-\Delta}, 0, 0\right) - \Gamma' G\left(\sqrt{-\Delta/4}\right)\,, & \Delta < 0 \\ I\left(\sqrt{\frac{\Delta}{3}}, \sqrt{\frac{\Delta}{3}}, \sqrt{\frac{\Delta}{3}}\right) - 3\Gamma' G\left(\sqrt{\frac{\Delta}{12}}\right)\,, & \Delta > 0\,, \end{cases}$$

where

$$G(w) = \int_0^w \frac{\mathrm{d}s}{(s^2+\Delta/4)^\varepsilon} \tag{A.14}$$

may be integrated in terms of Gauss hypergeometric functions. Whereas the diagram $I(x,0,0)$ is easily evaluated using elementary means, the equal mass diagram is not so simple. To proceed we may either use the explicit form of $I(x,x,x)$ given in the literature, e.g. [35, 64], or analytically continue the result for $\Delta < 0$. Either way we find

$$I(x,y,z) = \sin \pi\varepsilon \, I\left(\sqrt{\Delta},0,0\right) + \Gamma' \left[G\left(\frac{c}{2} - x\right) + \text{cycl.}\right] , \tag{A.15}$$

a result that holds for arbitrary $\Delta(x,y,z)$. The square root, $\sqrt{\Delta} = \exp(1/2 \log \Delta)$, is always taken on its principle branch.

Finally we examine the expansion of $I(x,y,z)$ around $d=4$. This is found by using

$$I(x,0,0) = -\frac{x}{(4\pi)^4}\left(\frac{4\pi\mu^2}{x}\right)^{2\varepsilon} \Gamma(\varepsilon-1)\Gamma(2\varepsilon-1)\Gamma(1-\varepsilon) \tag{A.16}$$

and expanding the denominator in the integrand of eq. (A.14). The result is

$$(4\pi)^4 I(x,y,z) = -\frac{c}{2\varepsilon^2} + \frac{\hat{L}_1}{\varepsilon} - 1/2\left(\mathrm{c}\left(\zeta(2)+\frac{5}{2}\right) + 2\hat{\mathrm{L}}_2 + \tilde{\xi}(\mathrm{x,y,z})\right) + \mathrm{O}(\varepsilon) \, , \tag{A.17}$$

where we've used the "natural" renormalisation point[8] $\hat{\mu}^2 = 4\pi\mu^2 \mathrm{e}^{3/2-\gamma}$,

$$\hat{L}_n := x \log^n \frac{x}{\hat{\mu}^2} + y \log^n \frac{y}{\hat{\mu}^2} + z \log^n \frac{z}{\hat{\mu}^2} \, ,$$

and $\tilde{\xi}(x,y,z)$ decomposes as

$$\tilde{\xi}(x,y,z) = \xi(x,y,z) - [x \log(y/x)\log(z/x) + \text{cycl.}] \tag{A.18}$$

$$\xi(x,y,z) = 2\sqrt{\Delta}\left(N(2\theta_x) + N(2\theta_y) + N(2\theta_z)\right) \, . \tag{A.19}$$

Note that the above definition of $\xi(x,y,z)$ holds for all $x,y,z \geq 0$, in distinction to the separate definitions given in [33] for $\Delta > 0$ and $\Delta < 0$. In the previous equation we have used the function

$$N(\theta) = -\int_0^\theta \mathrm{d}\phi \log\left(2\cos\frac{\phi}{2}\right) \, , \tag{A.20}$$

that is related to the Lobachevsky function (and thus the dilogarithm and Clausen function [65]). It is evaluated on the angles

$$\theta_x = \arctan\frac{-x+y+z}{\sqrt{\Delta}} \quad \text{and cyclic} \, . \tag{A.21}$$

For $\Delta > 0$ the above angles are real and less than $\pi/2$, so that the function eq. (A.20) is equivalent to the log-cosine function, defined through the log-sine function (see e.g. [65, 66])

$$\mathrm{Lc}_j(\theta) = \mathrm{Ls}_j(\pi) - \mathrm{Ls}_j(\pi-\theta) \, , \quad \mathrm{Ls}_j(\theta) = -\int_0^\theta \mathrm{d}\phi \log^{j-1}\left|2\sin\frac{\phi}{2}\right| \, .$$

[8] This renormalisation point is only natural when we work with the graph that has not had its subdivergent graphs subtracted.

Although $\mathrm{Ls}_2(\theta) = \mathrm{Cl}_2(\theta)$, it is the log-sine series of functions rather than the Clausen series that gives the simplest ε-expansion for $\Delta > 0$ [39, 67].

The above results may be seen to be equivalent with those in [33] by shifting the renormalisation point to $\bar{\mu}^2 = \hat{\mu}^2 \mathrm{e}^{-3/2}$ so that

$$\hat{L}_2 = L_2 - 3L_1 + \frac{9}{4}c\,, \quad \text{with} \quad L_n := x \log^n \frac{x}{\bar{\mu}^2} + \text{cycl.}$$

and then replacing one of the L_2's using

$$\left[(x-y-z)\log\frac{y}{\bar{\mu}^2}\log\frac{z}{\bar{\mu}^2} + \text{cycl.}\right] = L_2 - \left[x\log\frac{y}{x}\log\frac{z}{x} + \text{cycl.}\right] \,.$$

Finally the form of $\xi(x,y,z)$ can be seen to agree with that in [33] by noting

$$\theta_x + \theta_y + \theta_z = \mathrm{sgn}(\Delta)\frac{\pi}{2}$$

and, in the case of $\Delta < 0$, rewriting their ϕ_w in terms of θ_w using the standard formula [68]

$$\mathrm{arcoth}(z) = \mathrm{i}\left(\arctan\left(\frac{z}{\mathrm{i}}\right) \pm \frac{\pi}{2}\right)\,, \quad \pm \text{ if } \quad \pm z > 1\,.$$

It is also straightforward to see, in the case $\Delta > 0$ where

$$\theta_x = \frac{\pi}{2} - \arccos\frac{-x+y+z}{\sqrt{4yz}} \text{ and cyclic}\,,$$

that we obtain the result of [34, 39],[9] written in terms of log-sine functions and that our expression is automatically the correct analytic continuation for $\Delta < 0$.[10] It would be interesting to see if rewriting the results of [39] in terms of the angles eq. (A.21) would provide the correct analytic continuation to $\Delta < 0$ at all orders in the epsilon expansion.

In conclusion it is interesting to compare the above form of $\tilde{\xi}(x,y,z)$ with that obtained from the method of Veltman and van der Bij [29] (see also [30, 31]). Their method also leads to a single expression that holds for all x, y and z, but is better at revealing the simple mass dependence of $\tilde{\xi}$. Explicitly we find

$$\tilde{\xi}(x,y,z) = xf\left(\frac{y}{x},\frac{z}{x}\right) + yf\left(\frac{z}{y},\frac{x}{y}\right) + zf\left(\frac{x}{z},\frac{y}{z}\right) \tag{A.22}$$

$$f(a,b) = \int_0^1 \mathrm{d}\alpha \left(\mathrm{Li}_2(1-w) + \frac{w\log w}{w-1}\right)\,, \quad w = \frac{a}{\alpha} + \frac{b}{1-\alpha}\,, \tag{A.23}$$

where the integral in eq. (A.23) can be performed in terms of dilogarithms [29–31]. We note that Δ naturally appears during this integration.

[9] The latter paper provides an all order ε-expansion of $I(x,y,z)$ starting from its hypergeometric representation (eq. (A.15) and [33, 34]) and the "magic connection" [37].

[10] The procedure for analytic continuation advocated in [39, 67] involves rewriting the log-sine integrals in terms of the generlised Neilson polylogarithms, a nontrivial task at higher orders.

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